

# Periodic and rapid decay rank two self-adjoint commuting differential operators

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*Dedicated to Sergey Petrovich Novikov on his 75th birthday*

**ABSTRACT.** Self-adjoint rank two commuting ordinary differential operators are studied in this paper. Such operators with trigonometric, elliptic and rapid decay coefficients corresponding to hyperelliptic spectral curves are constructed. Some problems related to the Lamé operator and rank two solutions of soliton equations are discussed.

## 1 Introduction

Almost all solutions of soliton equations obtained in the last decades with the help of finite-gap theory are rank one solutions. This means that eigenfunctions of auxiliary linear operators form linear bundles over spectral curves. Meanwhile in the outstanding papers of I.M. Krichever and S.P. Novikov [1],[2] rank  $l > 1$  solutions are investigated (i.e. the eigenfunctions form vector bundles of rank  $l$  over the spectral curves). One of the main difficulty to construct higher rank solutions (even numerically) is the problem of finding higher rank commuting operators.

In [3] an example of rank two operators corresponding to nonsingular spectral curves of arbitrary genus is constructed. The operator

$$L_4^\sharp = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g+1)\alpha_3 x, \quad \alpha_3 \neq 0$$

commutes with a differential operator  $L_{4g+2}^\sharp$  of order  $4g+2$  [3]. The spectral curve is hyperelliptic curve of genus  $g$ . In this paper we further develop methods of [3] and construct self-adjoint operators of rank two with trigonometric, elliptic and rapid decay coefficients corresponding to hyperelliptic spectral curves. It is remarkable that there are higher rank operators with periodic and rapid decay coefficients expressed in terms of elementary functions (see Corollaries 1 and 2 below). The main results of this paper are Theorems 1 and 2.

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**Theorem 1** *The operator*

$$L_4^{\natural} = (\partial_x^2 + \alpha_1 \mathcal{P}(x) + \alpha_0)^2 + \alpha_1 g_2 g(g+1) \mathcal{P}(x), \quad \alpha_1 \neq 0,$$

where  $\mathcal{P}$  satisfies the equation

$$(\mathcal{P}'(x))^2 = g_2 \mathcal{P}^2(x) + g_1 \mathcal{P}(x) + g_0, \quad g_2 \neq 0$$

commutes with an operator  $L_{4g+2}^{\natural}$  of order  $4g+2$ .

At  $g_0 = 1, g_1 = 0, g_2 = -1$  we have

**Corollary 1** *The operator with periodic coefficients*

$$L_4^{\natural} = (\partial_x^2 + \alpha_1 \cos(x) + \alpha_0)^2 - \alpha_1 g(g+1) \cos(x), \quad \alpha_1 \neq 0$$

commutes with an operator of order  $4g+2$ .

Let  $\wp(z)$  be the Weierstrass elliptic function satisfying the equation

$$(\wp'(z))^2 = 4\wp^3(z) + g_2 \wp^2(z) + g_1 \wp(z) + g_0.$$

**Theorem 2** *The operator*

$$L_4^{\natural} = (\partial_x^2 + \alpha_1 \wp(x) + \alpha_0)^2 + s_1 \wp(x) + s_2 \wp^2(x),$$

where

$$\alpha_1 = \frac{1}{4} - 2g^2 - 2g, \quad s_1 = \frac{1}{4}g(g+1)(16\alpha_0 + 5g_2), \quad s_2 = -4g(g+2)(g^2 - 1),$$

$\alpha_0$  is an arbitrary constant, commutes with an operator of order  $4g+2$ .

At  $g_2 = 4a^2$  we have

**Corollary 2** *The operator with rapid decay coefficients*

$$L_4^{\natural} = \left( \partial_x^2 - \frac{\alpha_1 a^2}{\cosh^2(ax)} + \alpha_0 \right)^2 - \frac{s_1 a^2}{\cosh^2(ax)} + \frac{s_2 a^4}{\cosh^4(ax)}, \quad a \neq 0$$

where

$$\alpha_1 = \frac{1}{4} - 2g^2 - 2g, \quad s_1 = \frac{1}{4}g(g+1)(16\alpha_0 + 20a^2), \quad s_2 = -4g(g+2)(g^2 - 1),$$

$\alpha_0$  is an arbitrary constant, commutes with an operator of order  $4g+2$ .

Equations of spectral curves for pairs  $L_4^{\natural}, L_{4g+2}^{\natural}$  and  $L_4^{\natural}, L_{4g+2}^{\natural}$  are given in the Lemmas 1 and 2.

In the section 2 we recall the method of deformation of the Tyurin parameters. In the section 3 we prove Theorems 1 and 2. In the Appendix I the spectral curve and eigenfunctions of the Lamé operator

$$L_2 = -\partial_x^2 + g(g+1)\wp(x)$$

are found in some special form. In the Appendix II rank two solutions of soliton equations are discussed.

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## 2 Krichever–Novikov theory (method of deformation of the Tyurin parameters)

In this section we discuss some results on higher rank commuting ordinary differential operators and recall the method of deformation of Tyurin parameters (applications of this method for finding higher rank solutions of soliton equations see in [1],[2],[4]).

Commutative rings of ordinary differential operators were classified by I.M. Krichever [5]. Common eigenvalues of commuting operators

$$L_n = \partial_x^n + \sum_{i=0}^{n-2} u_i(x) \partial_x^i, \quad L_m = \partial_x^m + \sum_{i=0}^{m-2} v_i(x) \partial_x^i$$

are parametrized by points of the spectral curve  $\Gamma$ . If

$$L_n \psi = z \psi, \quad L_m \psi = w \psi, \quad (1)$$

then  $(z, w) \in \Gamma$ , where  $\Gamma$  is the smooth compactification of the curve defined by the equation  $R(z, w) = 0$ ,  $R$  is the Burchall–Chaundy [6] polynomial such that  $R(L_n, L_m) = 0$ . The rank of the pair  $(L_n, L_m)$  is called the dimension of the space of common eigenfunctions (1) for fixed  $P = (z, w) \in \Gamma$  in general position. Eigenfunctions (Baker–Akhiezer function) and coefficients of rank one operators can be expressed via theta-function of  $\Gamma$  [7]. Operators of rank two corresponding to the elliptic spectral curves were found by I.M. Krichever and S.P. Novikov [1], operators of rank three with the same spectral curve were found by O.I. Mokhov [4]. These operators and some related problems were studied in the papers [10]–[8]. Some examples of operators of rank 2 and 3 with spectral curves of genus 2,3,4 were found in [18]–[21].

The case of rank 1 the Baker–Akhiezer function  $\psi(x, P)$  has zeros divisor  $\gamma(x) = \gamma_1(x) + \dots + \gamma_g(x)$  where  $g$  is genus of  $\Gamma$ . The  $x$ -trajectory of  $\gamma$  in the Jacobi variety of  $\Gamma$  is a straight line. In the case of higher rank corresponding  $x$ -dynamics in the moduli space of vector bundles over  $\Gamma$  is very complicated. Let me recall spectral properties of  $\psi = (\psi_1, \dots, \psi_l)$  at  $l > 1$  [5]. The Baker–Akhiezer function  $\psi$  has  $lg$  simple poles  $p_1, \dots, p_{lg}$  with the properties

$$\text{Res}_{p_i} \psi_j = v_{i,j} \text{Res}_{p_i} \psi_l, \quad 1 \leq i \leq lg, \quad 1 \leq j \leq l-1.$$

The set

$$\{(p_j, v_{ik})\} \quad (2)$$

is called *Tyurin parameters*, which define a holomorphic bundle of rank  $l$  over  $\Gamma$ . There is a point  $q \in \Gamma$  where  $\psi$  has essential singularity of the form

$$\psi = \left( \sum_{s=0}^{\infty} \xi_s(x) k^{-s} \right) \Psi(x, k),$$

where  $\xi_0 = (1, 0, \dots, 0)$ ,  $\xi_i(x) = (\xi_i^1(x), \dots, \xi_i^l(x))$ ,  $k^{-1}$  is a local parameter, the matrix  $\Psi$  satisfies the equation

$$\frac{d\Psi}{dx} = A\Psi, \quad A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ k + \omega_0(x) & \omega_1(x) & \omega_2(x) & \dots & \omega_{l-2}(x) & 0 \end{pmatrix}.$$

The function  $\psi$  can not be found explicitly. To find operators one can use the following *method of deformation of Tyurin parameters*. Let us consider for simplicity the case of rank two. Then for  $P \in \Gamma$  the space of common eigenfunctions satisfies the second order differential equation

$$\psi'' - \chi_1(x, P)\psi' - \chi_0(x, P)\psi = 0.$$

The operator  $\partial_x^2 - \chi_1 \partial_x - \chi_0$  is a common right divisor of  $L_n - z$  and  $L_m - w$ . Functions  $\chi_0, \chi_1$  are rational functions on  $\Gamma$ , with poles at  $p_1(x), \dots, p_{2g}(x)$  and  $\chi_0$  has additional pole at  $q$ ,  $\chi_1$  has zero at  $q$

$$\chi_0 = k + O(1), \quad \chi_1 = O(1/k).$$

Let  $k - \gamma_i(x)$  be a local parameter near  $p_i(x)$ , then

$$\chi_0(x, P) = \frac{-v_{i,0}(x)\gamma_i'(x)}{k - \gamma_i(x)} + d_{i,0}(x) + O(k - \gamma_i(x)),$$

$$\chi_1(x, P) = \frac{-\gamma_i'(x)}{k - \gamma_i(x)} + d_{i,1}(x) + O(k - \gamma_i(x)),$$

where  $d_{i,0}, d_{i,1}, \gamma_i(x), v_{i,0}(x)$  satisfy the Krichever equations [5]

$$d_{i,0}(x) = v_{i,0}^2(x) + v_{i,0}(x)d_{i,1}(x) - v_{i,0}'(x), \quad i = 1, \dots, 2g. \quad (3)$$

The last equations define deformations of Tyurin parameters and  $(p_i(x), v_{i,0}(x))$  coincides with (2) at some  $x = x_0$ . I.M. Krichever and S.P. Novikov found  $\chi_0, \chi_1$  and corresponding operators at  $g = 1, l = 2$ .

**Theorem 3** (*I.M. Krichever, S.P. Novikov [1]*) *The operator*

$$\begin{aligned} L_{KN} &= (\partial_x^2 + u)^2 + 2c_x(\wp(\gamma_2) - \wp(\gamma_1))\partial_x + (c_x(\wp(\gamma_2) - \wp(\gamma_1)))_x - \wp(\gamma_2) - \wp(\gamma_1), \\ \gamma_1(x) &= \gamma_0 + c(x), \quad \gamma_2(x) = \gamma_0 - c(x), \\ u(x) &= -\frac{1}{4c_x^2} + \frac{1}{2}\frac{c_{xx}^2}{c_x^2} + 2\Phi(\gamma_1, \gamma_2)c_x - \frac{c_{xxx}}{2c_x} + c_x^2(\Phi_c(\gamma_0 + c, \gamma_0 - c) - \Phi^2(\gamma_1, \gamma_2)), \\ \Phi(\gamma_1, \gamma_2) &= \zeta(\gamma_2 - \gamma_1) + \zeta(\gamma_1) - \zeta(\gamma_2), \end{aligned}$$

where  $c(x)$  is an arbitrary smooth function,  $\gamma_0$  is a constant, commutes with an operator of order six.

The very interesting examples of the Krichever–Novikov operators are Dixmier operators.

**Theorem 4** (*J. Dixmier [22]*) *The operators*

$$\begin{aligned} L_D &= \left( \frac{d^2}{dx^2} + x^3 + h \right)^2 + 2x, \\ \tilde{L}_D &= \left( \frac{d^2}{dx^2} + x^3 + h \right)^3 + \frac{3}{2} \left( x \left( \frac{d^2}{dx^2} + x^3 + h \right) + \left( \frac{d^2}{dx^2} + x^3 + h \right) x \right) \end{aligned}$$

commute, herewith

$$L_D^3 = \tilde{L}_D^2 - h.$$

The action of the automorphisms group of the first Weyl algebra on the Dixmier operators and on the operators  $L_4^{\natural}, L_{4g+2}^{\natural}$  were studied in [23]. The conditions when  $L_{KN}$  has rational coefficients were found in [9].

**Theorem 5** (*P.G. Grinevich [9]*) *The operator  $L_{KN}$  has rational coefficients if and only if*

$$c(x) = \int_{q(x)}^{\infty} \frac{dt}{\sqrt{P(t)}},$$

where  $q(x)$  is a rational function. If  $\gamma_0 = 0$  and  $q(x) = x$ , then the operator  $L_{KN}$  coincides with the Dixmier operator  $L_D$ .

The conditions when  $L_{KN}$  is self-adjoint were found in [10].

**Theorem 6** (*P.G. Grinevich, S.P. Novikov [10]*) *Operator  $L_{KN}$  is formally self-adjoint if and only if  $\wp(\gamma_1) = \wp(\gamma_2)$ .*

In [3] the operators of rank two corresponding to the hyperelliptic spectral curve

$$\Gamma : w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \cdots + c_0$$

were considered

$$L_4\psi = z\psi, \quad L_{4g+2}\psi = w\psi, \quad P = (z, w) \in \Gamma.$$

**Theorem 7** (M. [3]) *The operator  $L_4$  is self-adjoint if and only if*

$$\chi_1(x, P) = \chi_1(x, \sigma(P)), \quad (4)$$

where  $\sigma$  is the involution

$$\sigma(z, w) = (z, -w).$$

Let us assume that  $\chi_1$  is invariant under  $\sigma$ , then  $L_4$  can be represented as

$$L_4 = (\partial_x^2 + V(x))^2 + W(x)$$

and  $\chi_0, \chi_1$  have the form

**Theorem 8** (M. [3]) *Functions  $\chi_0, \chi_1$  have the form*

$$\chi_0 = -\frac{1}{2} \frac{Q''}{Q} + \frac{w}{Q} - V, \quad \chi_1 = \frac{Q'}{Q},$$

where  $Q$  is a polynomial in  $z$  of degree  $g$  with coefficients depending on  $x$

$$Q(x, z) = (z - \gamma_1(x)) \dots (z - \gamma_g(x)). \quad (5)$$

The polynomial  $Q$  satisfies the equation

$$4F_g = 4(z - W)Q^2 - 4V(Q')^2 + (Q'')^2 - 2Q'Q^{(3)} + 2Q(2V'Q' + 4VQ'' + Q^{(4)}), \quad (6)$$

where  $Q', Q'', Q^{(k)}$  mean  $\partial_x Q, \partial_x^2 Q, \partial_x^k Q$ .

**Corollary 3** *The function  $Q$  satisfies the linear equation*

$$\mathcal{L}_5 Q = (\partial_x^5 + 2\langle V, \partial_x^3 \rangle + 2\langle z - W - V'', \partial_x \rangle) Q = 0, \quad (7)$$

where  $\langle A, B \rangle = AB + BA$ , the operator  $\mathcal{L}_5$  is skew-symmetric. Potentials  $V, W$  have the form

$$V = \left( \frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2} \right) \Big|_{z=\gamma_j}, \quad (8)$$

$$W = -2(\gamma_1 + \dots + \gamma_g) - c_{2g}.$$

The functions  $\gamma_1(x), \dots, \gamma_g(x)$  satisfy the equations

$$\frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2} \Big|_{z=\gamma_j} = \frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2} \Big|_{z=\gamma_k}. \quad (9)$$

So, Theorem 8 and Corollary 3 show that the spectral theory of the operator  $L_4$  is similar to the spectral theory of the finite-gap one dimensional Schrödinger operator (see [24])

$$(-\partial_x^2 + u(x))\psi = z\psi.$$

To the Schrödinger operator also corresponds a skew-symmetric operator but of order three [24] (see also [25])

$$\mathcal{L}_3 Q = (\partial_x^3 + 2\langle z - u, \partial_x \rangle) Q = 0,$$

and  $Q$  satisfies the nonlinear equation

$$4F_g(z) = 4(z - u)Q^2 - (Q')^2 + QQ'', \quad (10)$$

The potential  $u$  has the form

$$u = -2(\gamma_1 + \dots + \gamma_g) - c_{2g}.$$

Equations (9) are analogues of the Dubrovin equations

$$\gamma'_j = \pm \frac{2i\sqrt{F_g(\gamma_j)}}{\prod_{k \neq j} (\gamma_k - \gamma_j)}.$$

Using the equation (6) V.N. Davletshina [26] proved that the operator

$$(\partial_x^2 + \alpha_1 e^x + \alpha_0)^2 + g(g+1)\alpha_1 e^x, \quad \alpha_1 \neq 0$$

commutes with an operator of order  $4g+2$ .

## 3 Proof of Theorems

### 3.1 Theorem 2

We construct the polynomial  $Q^{\natural}$  satisfying the equation (7) for

$$V^{\natural} = \alpha_1 \mathcal{P}(x) + \alpha_0, \quad W^{\natural} = \alpha_1 g_2 g(g+1) \mathcal{P}(x).$$

Let

$$Q^{\natural}(x, z) = A_g^{\natural}(z) \mathcal{P}^g(x) + \dots + A_0^{\natural}(z).$$

Let us substitute  $Q^{\natural}$  into (7) and use the identities

$$\mathcal{P}''(x) = g_2 \mathcal{P}(x) + \frac{g_1}{2}, \quad \mathcal{P}'''(x) = g_2 \mathcal{P}'(x),$$

$$\mathcal{P}^{(4)}(x) = g_2^2 \mathcal{P}(x) + \frac{g_1 g_2}{2}, \quad \mathcal{P}^{(5)}(x) = g_2^2 \mathcal{P}'(x).$$

We get

$$-\frac{1}{4}\mathcal{P}'(x)(\beta_g\mathcal{P}^g(x) + \cdots + \beta_0) = 0,$$

where

$$\begin{aligned} \beta_s = & 4A_{s+5}g_0^2\frac{(s+5)!}{s!} + 4A_{s+4}g_0g_1\frac{(s+4)!}{s!}(2s+5) + A_{s+3}\frac{(s+3)!}{s!}(16\alpha_0g_0 + g_1(2s+3)(2s+5) \\ & + 8g_0g_2(s(s+4) + 5) + 2A_{s+2}\frac{(s+2)!}{s!}(2s+3)(4\alpha_1g_0 + g_1(4\alpha_0 + g_2(2s(s+3) + 5))) + \\ & 4A_{s+1}(s+1)((s+1)^2(4\alpha_1g_1 + g_2(4\alpha_1g_1 + g_2(4\alpha_0 + g_2(s+1)^2 + 4z) + \\ & 8A_s(2s+1)\alpha_1g_2(s(s+1) - g(g+1))) = 0, \end{aligned}$$

we assume that  $A_s = 0$  at  $s < 0$  and  $s > g$ . Hence from  $\beta_s = 0, 0 \leq s < g$  we have

$$\begin{aligned} A_s = & \frac{-1}{8(2s+1)\alpha_1g_2(s(s+1) - g(g+1))} \left( 4A_{s+5}g_0^2\frac{(s+5)!}{s!} + 4A_{s+4}g_0g_1\frac{(s+4)!}{s!}(2s+5) \right. \\ & + A_{s+3}\frac{(s+3)!}{s!}(16\alpha_0g_0 + g_1(2s+3)(2s+5) + 8g_0g_2(s(s+4) + 5)) + \\ & \left. + 4A_{s+1}(s+1)((s+1)^2(4\alpha_1g_1 + g_2(4\alpha_1g_1 + g_2(4\alpha_0 + g_2(s+1)^2 + 4z))) \right). \end{aligned} \quad (11)$$

Let

$$A_g = \frac{(1 \cdot 3 \cdot \dots \cdot (2g+1)\alpha_1^g(2 - g(g+1)) \dots ((g-1)g - g(g+1)))}{4^gg!}. \quad (12)$$

Then  $Q^\natural$  has the form (5) and satisfies (7) and (6). Theorem 2 is proved.

Let us find the spectral curve.

**Lemma 1** *The spectral curve of  $L_4^\natural, L_{4g+2}^\natural$  is given by the equation*

$$\begin{aligned} w^2 = & \frac{1}{4}(4A_0^2z + A_0(16\alpha_0g_0A_2 + 48g_0^2A_4 + 36g_0g_1A_3 + 3g_1^2A_2 + 16g_0g_2A_2 + A_1((25 - 8g(g+1))g_0 \\ & + g_1(4\alpha_0 + g_2))) - 4\alpha_0g_0A_1^2 + \frac{1}{4}(4g_0A_2 + g_1A_1)^2 - 2g_0A_1(6g_0A_3 + 3g_1A_2 + g_2A_1)), \end{aligned}$$

where  $A_j$  are defined in (11), (12).

*Proof.* Let  $x_0$  be a zero of  $\mathcal{P}(x)$ . Since in the right hand side of (6) there are only derivatives of order not greater than four, then we have

$$4\tilde{F}_g = \frac{1}{4}(4(z - W)\tilde{Q}^2 - 4V(\tilde{Q}')^2 + (\tilde{Q}'')^2 - 2\tilde{Q}'\tilde{Q}^{(3)} + 2\tilde{Q}(2V'\tilde{Q}' + 4V\tilde{Q}'' + \tilde{Q}^{(4)}))|_{x=x_0},$$

where

$$\tilde{Q}^\natural = A_4^\natural(z)\mathcal{P}^4(x) + \cdots + A_0^\natural(z).$$

Using

$$(\mathcal{P}'(x))^2 = g_0, \quad \mathcal{P}''(x) = \frac{g_1}{2}, \quad \mathcal{P}^{(3)}(x) = g_2\mathcal{P}'(x), \quad \mathcal{P}^{(4)}(x) = \frac{g_1g_2}{2}$$



we get the equation. Lemma 1 is proved.

Let

$$H^{\natural} = \partial_x^2 + \alpha_1 \cos(x) + \alpha_0, \quad L_4^{\natural} = (H^{\natural})^2 - \alpha_1 g(g+1) \cos(x)$$

and we shall use the notation

$$< A, B > = AB + BA.$$

**Examples:**

1)  $g = 1$

$$L_6^{\natural} = (H^{\natural})^3 + \frac{1}{8} < 1 - 4\alpha_0 - 12\alpha_1 \cos(x), H^{\natural} > + \alpha_1 \cos(x),$$

$$F_1^{\natural}(z) = z^3 + \left(\frac{1}{2} - 2\alpha_0\right)z^2 + \frac{1}{16}(1 - 8\alpha_0 + 16\alpha_0^2 - 16\alpha_1^2)z + \frac{\alpha_1^2}{4}.$$

2)  $g = 2$ , let  $\alpha_0 = 0$

$$L_{10}^{\natural} = (H^{\natural})^5 + \frac{1}{2} < \frac{17}{4} - 15\alpha_1 \cos(x), (H^{\natural})^3 > - \frac{15}{2}\alpha_1 < \cos(x), (H^{\natural})^2 > +$$

$$\frac{1}{2} < 1 + 27\alpha_1^2 - 60\alpha_1 \cos(x) + 45\alpha_1^2 \cos(2x), H^{\natural} > + \frac{3}{2}\alpha_1(5 \cos(x) - 12\alpha_1),$$

$$F_2^{\natural}(z) = z^5 + \frac{17}{2}z^4 + \frac{1}{16}(321 - 336\alpha_1^2)z^3 + \frac{1}{4}(34 - 531\alpha_1^2)z^2 + \\ (1 - 189\alpha_1^2 + 108\alpha_1^4)z + 24\alpha_1^2 + 513\alpha_1^4.$$

### 3.2 Theorem 3

Let

$$Q^b(x, z) = A_g^b(z)\wp^g(x) + \cdots + A_0^b(z).$$

Let us substitute  $Q^b$  into (7). Using the identities

$$\wp''(x) = (6\wp^2(x) + g_2\wp(x) + \frac{g_1}{2}), \quad \partial_x^3 \wp(x) = (12\wp(x) + g_2)\wp'(x),$$

$$\partial_x^4 \wp(x) = (120\wp^3(x) + 30g_2\wp^2(x) + (18g_1 + g_2^2)\wp(x) + \frac{g_1g_2}{2} + 12g_0),$$

$$\partial_x^5 \wp(x) = (360\wp^2(x) + 60g_2\wp(x) + 18g_1 + g_2^2)\wp'(x)$$

we get

$$\mathcal{L}_5 Q = \wp'(x)(\beta_{g+1}\wp^{g+1}(x) + \cdots + \wp(x)) = 0,$$

where  $\beta_{s+1} = 0$  is equivalent to

$$16A_s(s+1)((6+8\alpha_1)s + (19+4\alpha_1)s^2 + 16s^3 + 4s^4 - s_2) + \\ 8A_{s+1}(2s+3)(4\alpha_0(s+1)(s+2) + g_2(s+1)(s+2)(2s^2 + 6s + \alpha_1 + 5) - s_1) + \\ 4A_{s+2}(4\alpha_0g_2(s+3)^3 + g_2^2(s+2)^5 + 2g_1(s+2)^3(4(s+2)^2 + 2\alpha_1 + 5) + 4(s+2)z) +$$

$$\begin{aligned}
& 2A_{s+3}(s+3)(s+2)(2s+5)(4g_0(\alpha_1 + 2(s^2 + 5s + 9)) + g_1(4\alpha_0 + g_2(2s^2 + 10s + 13))) + \\
& A_{s+4}(s+2)(s+3)(s+4)(16\alpha_0g_0 + 8g_0g_2(s^2 + 6s + 10) + g_1^2(4s^2 + 24s + 35)) + \\
& 4A_{s+5}g_0g_1(s+2)(s+3)(s+4)(s+5)(2s+7) + 4A_{s+6}g_0^2\frac{(s+6)!}{(s+1)!} = 0.
\end{aligned}$$

At  $s = g$  and  $s = g - 1$  this formula has the forms

$$A_g(g+1)((6+8a_1)g + (19+4a_1)g^2 + 16g^3 + 4g^4 - s_2) = 0, \quad (13)$$

$$\begin{aligned}
& A_g(2g+1)(4a_0g(g+1) + g(g+1)(1+a_1+2g+2g^2)g_2 - s_1) + \\
& 2A_{g-1}g(1-5g^2+4g^4+4a_1(g^2-1) - s_2) = 0.
\end{aligned} \quad (14)$$

At

$$s_1 = \frac{1}{4}g(g+1)(16\alpha_0 + 5g_2), \quad s_2 = -4g(g+2)(g^2-1), \quad a_1 = \frac{1}{4} - 2g - 2g^2$$

equations (13),(14) are vanished. From  $\beta_{g-1} = 0$  we get

$$\begin{aligned}
A_{g-2} = & \frac{A_{g-1}(4\alpha_0(8g-4) + (16g^3 - 24g^2 + 18g - 5)g_2)}{64(2g^2 - 3g + 1)} + \\
& \frac{A_g(8g^3g_1 - g^4g_2^2 - g^2(11g_1 + 4\alpha_0g_2) - 4z)}{64(2g^2 - 3g + 1)}.
\end{aligned}$$

Thus from the equations  $\beta_i = 0, i > 0$  we express  $A_{i-1}$  through  $A_{g-1}$  and  $A_g$ . The last equation  $\beta_0 = 0$  takes the form

$$A_gP_0(z) + A_{g-1}P_1(z) = 0,$$

where  $P_0(z)$  and  $P_1(z)$  are some polynomials. Let

$$A_g = -P_1(z), \quad A_{g-1} = P_0(z).$$

So, we find polynomial in  $z$  solution of the equation (1). Theorem 3 is proved.

The equation  $w^2 = F_g^b(z)$  of the spectral curve of  $L_4^b, L_{4g+2}^b$  is given in the following lemma.

**Lemma 2** *The spectral curve of  $L_4^b, L_{4g+2}^b$  is given by the equation*

$$\begin{aligned}
w^2 = & \frac{1}{4}(4A_0^2z + A_0(4\alpha_1A_1g_0 + 48g_0^2 + 36A_3g_0g_1 + 3A_2g_1^2 + 4\alpha_0(4A_2g_0 + A_1g_1) + 16A_2g_0g_2 + \\
& + A_1g_1g_2) - 4a_0A_1^2g_0 + \frac{1}{4}(4A_2g_0 + A_1g_1)^2 - 2A_1g_0(6A_3g_0 + 3A_2g_1 + A_1g_2)),
\end{aligned}$$

where  $A_j$  are defined above.

The proof of the Lemma 2 is the same as the proof of the Lemma 1.

Let

$$H^\flat = \partial_x^2 + \alpha_1 \wp(x) + \alpha_0.$$

**Examples:**

For the simplification of the formulas we put  $g_2 = 0$ .

3)  $g = 1$

$$L_6^\flat = (H^\flat)^3 + \frac{3}{8} < g_1 + 2\alpha_0 \wp(x), H^\flat > + 2\alpha_0 g_1 + 32\alpha_0 \wp^2(x),$$

$$F_1^\flat = z^3 + \frac{3g_1}{2}z^2 + (9\alpha_0 g_0 + 4\alpha_0^2 g_1 + \frac{9}{16}g_1^2)z + 4\alpha_0^2 g_1^2 + \frac{27\alpha_0 g_0 g_1}{4} - 16\alpha_0^3 g_0,$$

4)  $g = 2, g_1 = 0$

$$\begin{aligned} L_{10}^\flat &= (H^\flat)^5 + < 30\alpha_0 \wp(x) - 12\wp^2(x), (H^\flat)^3 > + 15 < 16g_0 - 12\alpha_0 \wp^2(x) + 16\wp^3(x), (H^\flat)^2 > \\ &\quad + 9 < \alpha_0 g_0 - 1850g_0 \wp(x) + 80\alpha_0^2 \wp^2(x) - 480\alpha_0 \wp^3 - 11520\wp^2(x), H^\flat > - \\ &\quad 36(28\alpha_0^2 g_0 - 1151\alpha_0 g_0 \wp - 6946g_0 \wp^2(x) + 160\alpha_0^2 \wp^3(x) - 7520\alpha_0 \wp^4(x) - 30208\wp^5(x)), \\ F_2^\flat(z) &= z^5 - 387\alpha_0 g_0 z^3 + 27g_0(16\alpha_0^3 + 139g_0)z^2 + 12636\alpha_0^2 g_0^2 z - 243\alpha_0 g_0^2(64\alpha_0^3 + 637g_0). \end{aligned}$$

## 4 Appendix I. Spectral curve and eigenfunctions of the Lamé operator

S.P. Novikov proved [27] that if the periodic Schrödinger operator

$$-\partial_x^2 + u(x)$$

commute with an operator of odd order then it has finite number of gaps in its spectrum (the inverse statement were proved by B.A. Dubrovin [28]). Such operator is called finite-gap operator. Eigenfunctions of the finite-gap operator are expressed in terms of theta-function of the hyperelliptic spectral curve via Its formula [29]. The famous important finite-gap operator is the Lamé operator

$$L_2 = -\partial_x^2 + g(g+1)\wp(x).$$

It was introduced in 1837. Hermite and Halphen used the following ansatz for eigenfunctions of the Lamé operator

$$\begin{aligned} L_2 \psi &= z \psi, \\ \psi &= \sum_{j=1}^{g(g+1)/2} a_j(z, \alpha_j) \Phi(x, \alpha_j), \end{aligned}$$

where

$$\Phi(x, \alpha) = -\frac{\sigma(x - \alpha)}{\sigma(\alpha)\sigma(x)} e^{\zeta(\alpha)x},$$

$a_j(z, \alpha_j)$  is some function. With the help of this ansatz Hermite and Halphen studied the cases  $g = 2, 3$ . I.M. Krichever [30] used some modification of the ansatz to find elliptic solutions of the Kadomtsev–Petviashvili equation (see also [31]). M.-P. Grosset and A.P. Veselov [32] proved that coefficients of the equations for spectral curve of the Lamé operator can be found recurrently with the help of elliptic Bernoulli polynomials.

In the next theorem we find the equation of the spectral curve in the explicit form. For simplicity of the formulas we assume in this section  $g_2 = 0$ , i.e. the Weierstrass function satisfies the equation

$$(\wp'(z))^2 = 4\wp^3(z) + g_1\wp(z) + g_0.$$

Let us introduce the following polynomials in  $z$  with the help of recurrent formulas. Let  $A_s = 0$  at  $s > g$  and  $s < 0$ ,

$$A_g = \frac{4^g(g^2 + g) \cdot (g^2 + g - (s+1)s) \cdot (g^2 + g - (g-1)g)}{g!},$$

$$A_s = \frac{(s+1)(8A_{s+1}z + A_{s+2}g_1(s+2)(2s+3) + 2A_{s+3}g_0(s+2)(s+3))}{4(2s+1)(g^2 + g - s(s+1))}.$$

We also introduce the following function

$$Q = A_g\wp^g(x) + \dots + A_0. \quad (15)$$

**Theorem 9** *The spectral curve of the Lamé operator is given by the equation*

$$w^2 = \frac{1}{4}(4A_0^2z + A_0(4A_2g_0 + A_1g_1) - A_1^2g_0).$$

*The eigenfunctions of the Lamé operator are*

$$\psi = \sqrt{Q}e^{iw \int \frac{dx}{Q}}.$$

*Proof.* Our observation is that the solution of the equation (10) for the Lamé operator is (15). The proof is the same as proof of Theorems 1 and 2. To find eigenfunctions we observe that the eigenfunctions satisfies the equation

$$\psi' - i\chi_0\psi = 0, \quad \chi_0 = \frac{Q_x}{2iQ} + \frac{w}{Q}.$$

It follows from

$$-\partial_x^2 + g(g+1)\wp(x) = (-\partial_x - i\chi_0)(\partial - i\chi_0).$$

Theorem 9 is proved.

The last theorem gives very effective way to find spectral curve of the Lamé operator.

**Examples:**

5)  $g = 1$

$$F_1 = z^3 + \frac{g_1}{4}z - \frac{g_0}{4}.$$

6)  $g = 2$

$$F_2 = z^5 + \frac{21g_1}{4}z^3 + \frac{27}{4}g_0z^2 + \frac{27}{4}g_1^2z + \frac{81}{4}g_0g_1.$$

7)  $g = 3$

$$F_3 = z^7 + \frac{63}{2}g_1z^5 + \frac{297}{2}g_0z^4 + \frac{4185}{16}g_1^2z^3 + \frac{18225}{8}g_0g_1z^2 + \frac{3375}{16}(27g_0^2 + g_1^3).$$

## 5 Appendix II. Rank two self-adjoint operators and evolution soliton equations

Let us consider the following system of evolution equations

$$V_t = \frac{1}{4}(6VV_x + 6W_x + V_{xxx}), \quad W_t = \frac{1}{2}(-3VW_x - W_{xxx}). \quad (16)$$

This system is equivalent to the commutativity condition

$$[L_4, \partial_t - \mathcal{A}_3] = 0,$$

where

$$L_4 = (\partial_x^2 + V(x, t))^2 + W(x, t), \quad \mathcal{A}_3 = \partial_x^3 + \frac{3}{2}V(x, t)\partial_x + \frac{3}{4}V_x(x, t).$$

Following to the papers of I.M. Krichever and S.P. Novikov [1, 2] we call the solution  $V(x, t), W(t, x)$  the solution of rank  $l$  if for every  $t$  the operator  $L_4$  is included in a pair of commuting operators of rank  $l$  (here  $l = 1$  or  $l = 2$ ). Rank 1 solutions of (16) were found by V.G. Drinfeld and V.V. Sokolov [33]. It is a very natural question what is the evolution equation for  $Q$  under (16)?

**Theorem 10** (V.N. Davletshina, M.) *The polynomial*

$$Q = (z - \gamma_1(x, t)) \dots (z - \gamma_g(x, t))$$

*satisfies the equation*

$$Q_t = \frac{1}{2}(-3VQ_x - Q_{xxx}). \quad (17)$$

We shall give the proof of the Theorems 10 and 11 and related statements in an another publication.

The equation (17) gives a symmetry of the equation (6). At  $g = 1$  the formula (8) gives

$$V = \frac{-16F_1(\frac{1}{2}(-c_2 - W)) + W_{xx}^2 - 2W_xW_{xxx}}{4W_x^2},$$

where

$$w^2 = F_1(z) = z^3 + c_2 z^2 + c_1 z + c_0,$$

is the equation of the spectral curve, and the equation (17) is reduced to the famous Krichever–Novikov (KN) equation

$$W_t = \frac{48F_1(\frac{1}{2}(-c_2 - W)) - W_{xx}^2 + 2W_x W_{xxx}}{8W_x}.$$

KN equation plays an important role in the theory of rank two solutions of the Kadomtsev–Petviashvili (KP) equation at  $g = 1$ . It would be very interesting to study the equation (17) in the framework of rank two solutions of KP at  $g > 1$ .

**Theorem 11** (V.N. Davletshina, M.) *The system (16) has the following rank two solution corresponding to the elliptic spectral curve*

$$W(x, t) = -40\wp(bt + x)^2 - \frac{20}{21}(8b + 7g_2)\wp(bt + x) + c,$$

$$V(x, t) = -10\wp(bt + x) - \frac{2b}{21} - \frac{5g_2}{6},$$

where  $b, c$  — const.

Let us numerically study the Cauchy problem for (16) with initial data from Corollary 2 and with the rapid decay boundary conditions

$$V(x, 0) = -\frac{\alpha_1 a^2}{\cosh^2(ax)} + \alpha_0,$$

$$W(x, 0) = -\frac{s_1 a^2}{\cosh^2(ax)} + \frac{s_2 a^4}{\cosh^4(ax)},$$

$$V_x(\pm\infty, t) = V_{xx}(\pm\infty, t) = W_x(\pm\infty, t) = W_{xx}(\pm\infty, t) = 0.$$

The behavior of the solution looks like  $g$ -soliton equation of the Korteweg–de Vries equation (see Fig. 1 — 12).

It is an interesting problem to find exact solution of the Cauchy problem. It gives soliton deformations of the rank two self-adjoint commuting differential operators.

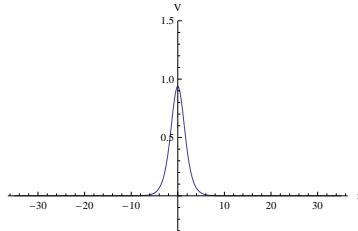


Figure 1: one-soliton solution,  $g = 1, a = 1/2, \alpha_0 = 0, t = 0$

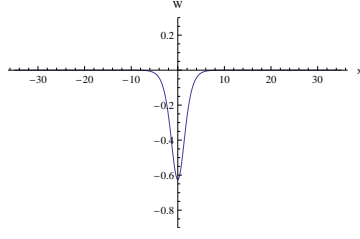


Figure 2: one-soliton solution,  $g = 1, a = 1/2, \alpha_0 = 0, t = 0$

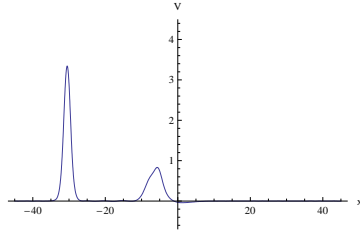


Figure 3: two-soliton solution,  $g = 2, a = 1/2, \alpha_0 = 0, t = -15$

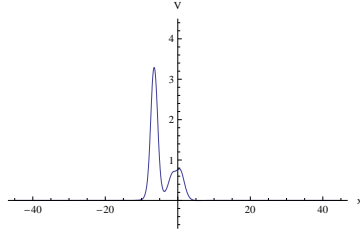


Figure 4: two-soliton solution,  $g = 2, a = 1/2, \alpha_0 = 0, t = -3$

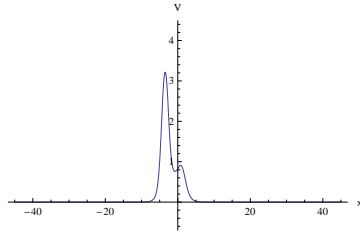


Figure 5: two-soliton solution,  $g = 2, a = 1/2, \alpha_0 = 0, t = -1.5$

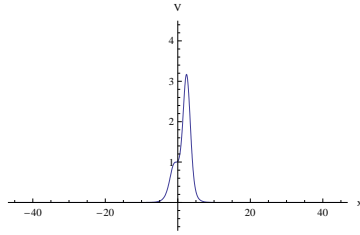


Figure 6: two-soliton solution,  $g = 2, a = 1/2, \alpha_0 = 0, t = 1$

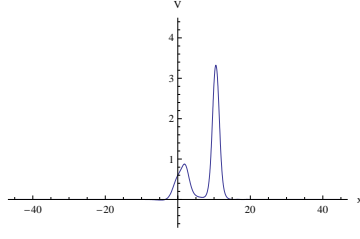


Figure 7: two-soliton solution,  $g = 2, a = 1/2, \alpha_0 = 0, t = 5$

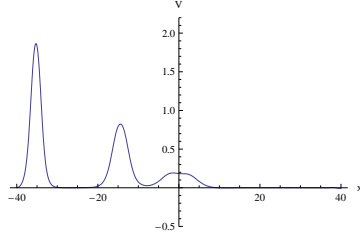


Figure 8: three-soliton solution,  $g = 3, a = 1/4, \alpha_0 = 0, t = -30$

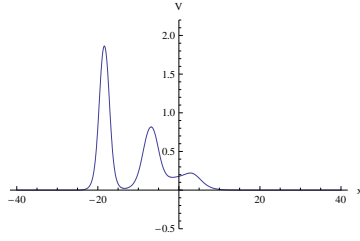


Figure 9: three-soliton solution,  $g = 3, a = 1/4, \alpha_0 = 0, t = -15$

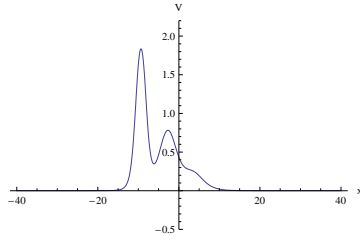


Figure 10: three-soliton solution,  $g = 3, a = 1/4, \alpha_0 = 0, t = -7$

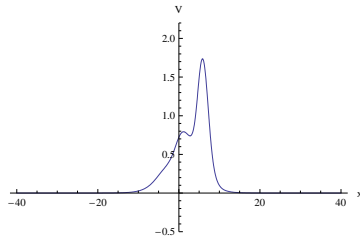


Figure 11: three-soliton solution,  $g = 3, a = 1/4, \alpha_0 = 0, t = 4$



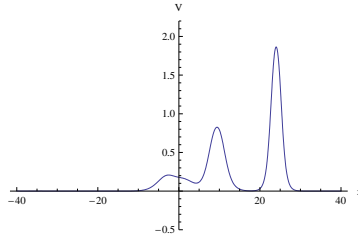


Figure 12: three-soliton solution,  $g = 3, a = 1/4, \alpha_0 = 0, t = 20$

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